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RD 89690
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By MBD Date 7/14/2025

ORTHOGONAL MOTION FOR CCM'S

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NSA-34
3 February 1956

Let η be a rotor device whose graph consists of ν cycles each of length λ. An orthogonal motion for η is a rule of motion giving λ cycles each of length ν, where each cycle of length ν consists of exactly one point from each cycle of length λ.

Suppose a CCM cascade of n wheels $w_1, w_2, \cdots, w_n$ has the property that the graph of $w_1, \cdots, w_k$, $k = 1, 2, \cdots, n$ consists of cycles all of equal length. An orthogonal motion for such a CCM cascade is defined and applied to a device discussed by Forrest S. Goepper (see "m'-Motion for a CCM", AFSA-412B6, 6 June 1952, C 25.221.1-503) to obtain an orthogonal motion for this device which is simpler than Goepper's m'-motion.

S-70 061

ORTHOGONAL MOTION FOR CCM's

1. Introduction

Let M be a rotor device whose graph consists of ν cycles $C_1, \cdots, C_\nu$ each of length λ for some positive integers ν, λ. An orthogonal motion for M is a rule of motion giving λ cycles $\Gamma_1, \cdots, \Gamma_\lambda$ each of length ν where Γ_i and C_j have exactly one point of intersection $(i = 1, \ldots, \nu;\ j = 1, \ldots, \lambda)$. An example is the m'-motion defined by Forrest S. Goepper, "M'-Motion for a CCM," (AFSA-412B6, 6 June 1952, C 25.221.1-503).

In the present report, an orthogonal motion is defined for a CCM cascade of N wheels $w_1, \ldots, w_N$ of respective lengths $m_1, \ldots, m_N$ where (1) w_1 is fast (2) w_i has a notch pattern controlling w_{i+1} in the usual way $(i = 1, \ldots, N-1)$ and (3) the graph of $(w_1, \ldots, w_k)$ consists of cycles all of equal length λ_k $(k = 1, \ldots, N)$. An orthogonal motion simpler than Goepper's m'-motion is obtained for the machine he discussed.

Let C^{k-1} be any cycle of $(w_1, \ldots, w_{k-1})$ and let $u(C^{k-1})$ be the number of points on C^{k-1} where a notch of w_{k-1} is active. Then

(1.1) $$\nu_k = d\left(m_k, u(C^{k-1})\right) \qquad \text{(d means g.c.d.)}$$

Condition (3) is equivalent to the requirement that ν_k be the same for every C^{k-1}. It is sufficient but not necessary that we have equal cycles for each k in order to have equal cycles at the final stage. We are assuming equality for each k in order to simplify our discussion.

2. ν-motion

Our orthogonal motion will be compounded of individual wheel motions which will be called ν-motions. Let w be a rotor of any length m, with its points numbered $0, 1, \ldots, m-1$ in cyclic order, and let ν be any factor of m.

By the ν-motion for w we will mean a motion such that, if $x-1$ is in active position, then the next point to come into active position will be x, unless x is a multiple of ν. In the latter case, $x-\nu$ will next come into active position. If, for example $m=12$, $\nu=4$ and $x=6$, the motion brings points $(6, 7, 4, 5, 6, 7, 4, 5, \cdots)$ successively into active position. If $x=9$, we would have $(9, 10, 11, 8, 9, 10, 11, 8, \cdots)$ and so on.

If m is even, 2-motion is defined and is a flipping back and forth between 2k and 2k+1 for any $k \in 0, 1, \cdots, \frac{m}{2} - 1$.

In accordance with the above definition, m-motion is the usual motion of a rotor in which the $x-1$ is always succeeded by

$x\,(x = 1, \ldots, m-1)$ and $m-1$ by 0. The "movement" of a stationary wheel satisfies the definition of <u>1-motion</u>.

The definition of ν-motion does not restrict the time interval between one step and the next, but merely specifies what the next step will be when it occurs.

3. <u>O-motion</u>

We refer to our orthogonal motion as <u>O-motion</u>, the O not being a zero but the initial letter of <u>orthogonal</u>. We proceed to define <u>O-motion</u>.

Given the N-wheel CCM described above, let C^{k-1} be any cycle of the first $(k-1)$ wheels $(k > 1)$. Then (C^{k-1}, w_k), regarded as a 2-wheel device, gives rise to

(3.1) $$\nu_k = \frac{\lambda_{k-1} m_k}{\lambda_k}$$

cycles of the first k wheels.

<u>(A) With the ν's thus defined, there are</u>

(3.2) $$\nu_{12\cdots k} = \nu_1 \cdot \nu_2 \cdots \nu_k$$

<u>cycles, each of length λ_k, in the graph of the k-wheel device $(w_1, \ldots, w_k)$. In particular,</u>

(3.3) $$\begin{cases} \nu_1 = 1 & \text{(hence } \lambda_1 = m_1\text{)} \\ \nu_{12\cdots k} = \nu_2 \nu_3 \cdots \nu_k \end{cases}$$

The statements made without proof in this report are simple consequences of known results in cycle theory. Some of the most relevant results are stated in the writer's Exhaustive motion for a CCM (NSA-34, 11 January 1956, S-70 038).

O-motion is a cascade of ν-motions where w_k has ν_k-motion and takes one step each time w_{k-1} steps onto a multiple of ν_{k-1}. Since $\nu_1 = 1$, the statement that w_1 has ν_1-motion means that it is, in effect, stationary or that it takes a trivial "step," with the same arbitrary but fixed point reappearing each unit of time. This means that w_2 has fast ν_2-motion, stepping each unit of time. Then w_3 has ν_3-motion, stepping each ν_{12} th unit of time; w_4 has ν_4-motion, stepping each ν_{123} th unit of time, and so on.

Theorem. O-motion is an orthogonal motion.

We establish this theorem by describing the O-motion cycles and the CCM-motion cycles in such a way as to reveal that they are related in the required manner.

(B) The CCM-cycle through $(0, x_2, \ldots, x_N)$ will be denoted by $C_{x_2 x_3 \cdots x_N}$ as the x's range independently through $x_j = 0, 1, \ldots, \nu_j - 1$ $(j = 2, \ldots, N)$. These cycles, which number

(3.4) $\quad \nu = \nu_2 \nu_3 \cdots \nu_N$

are of length

(3.5) $\lambda = \dfrac{m_1 m_2 \cdots m_N}{\nu}$

and constitute the entire CCM cyclic structure of the device. [See Section 5 below for further details.]

<u>(C) The O-motion cycle which passes through</u> $(j_1\nu_1, j_2\nu_2, j_3\nu_3, \cdots, j_N\nu_N) \equiv (j_1 \;\; , j_2\nu_2, j_3\nu_3, \cdots, j_N\nu_N)$ <u>will be denoted by</u> $\Gamma_{j_1 j_2 \cdots j_N}$ <u>as the j's range independently through</u>

$$j_i = 0, 1, \cdots, \frac{m_i}{\nu_i} - 1 \qquad (i = 1, \ldots, N).$$

Consider first the cycle $\Gamma_{oo\cdots o}$. By definition, its points are the ν points $(0, x_2, \ldots, x_N)$ $(x_j = 0, 1, \cdots, \nu_j - 1;\ j = 2, \ldots, N)$. To be precise, these points occur on $\Gamma_{o\cdots o}$ in lexicographic order with respect to their reversed coordinates $(x_N, x_{N-1}, \cdots, x_2)$. For example, if $N = 3$ and $\nu_2 = 4$, $\nu_3 = 3$ there would be twelve points on $\Gamma_{o\cdots o}$ in the order

$$\begin{array}{lllll} (0, 0, 0) \rightarrow & (0, 1, 0) & (0, 2, 0) & (0, 3, 0) & \\ (0, 0, 1) & (0, 1, 1) & (0, 2, 1) & (0, 3, 1) & \\ (0, 0, 2) & (0, 1, 2) & (0, 2, 2) & (0, 3, 2) & (0, 0, 0) \end{array}$$

In general, the O-cycle $\Gamma_{j_1 \cdots j_N}$ passes through the points $(j_1, j_2\nu_2 + x_2, j_3\nu_3 + x_3, \cdots, j_N\nu_N + x_N)$ in the same order that $\Gamma_{oo\cdots o}$ passes through the points $(0, x_2, \ldots, x_N)$. The number of cycles Γ is

(3.6) $\left(\dfrac{m_1}{\nu_1}\right)\left(\dfrac{m_2}{\nu_2}\right) \cdots \left(\dfrac{m_n}{\nu_n}\right) = \lambda$

and each is of length ν as required. Each setting of the machine is on one and only one $\Gamma_{j_1 \ldots j_N}$.

(D) No two of the ν points on $\Gamma_{j_1 \ldots j_N}$ are on the same cycle $C_{x_2 \ldots x_N}$. Hence each $\Gamma_{j_1 \ldots j_N}$ intersects each $C_{x_2 \ldots x_N}$ in a single point. [See Section 5(B) below.]

This completes a set of statements, easily verifiable in terms of known cycle theory, from which the above theorem follows.

The exhaustive motion defined in "Exhaustive Motion for a CCM" reduces, for the present case, to running around $C_{oo \ldots o}$, stepping one point along $\Gamma_{oo \ldots o}$, running around $C_{010 \ldots 0}$ stepping along $\Gamma_{oo \ldots o}$ and so on.

4. O-motion for a special case

The device discussed by Goepper (loc. cit.) consists of five 26-point rotors R_i $(i = 1, 2, 3, 4, 5)$, where (1) R_3 is fast and has a notch pattern on each side (2) R_3, R_4, R_5 form a CCM cascade using one of R_3's notch patterns (3) R_3, R_2, R_1 form a CCM cascade using the other of R_3's notch patterns. Restrictions on the notch patterns are such that each of the cascades R_3, R_4, R_5 and R_3, R_2, R_1 has four cycles, each of length $\lambda = 2 \cdot 13^3$. In particular, (R_3, R_2) produces two cycles each of length $2 \cdot 13^2$. Each of these, with R_1, produces two cycles of length λ.

(A) Applying the work of the previous section to the cascade R_3, R_2, R_1 alone, O-motion consists in a cascade of ν-motions, with R_3 stationary, R_2 undergoing 2-motion and R_1 undergoing 2-motion. In other words, R_2 flips back and forth between positions $2j$ and $2j+1$, stepping once per unit time; while R_1 flips back and forth between $2k$ and $2k+1$, stepping each alternate unit of time.

If $(x, 2j, 2k)$ are position numbers on (R_3, R_2, R_1) then O-motion takes us around the four-point cycle

Γ_{xjk}: $(x, 2j, 2k)$ $(x, 2j+1, 2k)$ $(x, 2j, 2k+1)$ $(x, 2j+1, 2k+1)$.

There is one of these four points on each of the cycles (C_1, C_2, C_3, C_4), though not generally in the order named.

<u>(B) As x ranges through $(0, 1, \cdots, 25)$ and $(j = 0, 1, \cdots, 12;\ k = 0, 1, \cdots, 12)$, Γ_{xjk} ranges through the $2 \cdot 13^3$ cycles of the O-motion structure. Each is of length 4 and meets each of the CCM cycles C_i $(i = 1, 2, 3, 4)$, which are of length $2 \cdot 13^3$, in a single point.</u>

Let (x, y, z) be any point on one of the cycles C_i and let (s, t) be any setting of the two rotors (R_4, R_5). The CCM cycle of (R_3, R_2, R_1) through (x, y, z) is of exactly the same length, $2 \cdot 13^3$, as the CCM cycle of (R_3, R_4, R_5).

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Hence the setting (x, y, z, s, t) will recur in $2 \cdot 13^3$ units of time.

<u>(C) The cyclic structure of the entire device consists of $4 \cdot 26^2 = 2^4 \cdot 13^2$ cycles C_{ist} $(i = 1, 2, 3, 4;\ s = 0, 1, \cdots, 25;$ $t = 0, 1, \cdots, 25)$ each of length $2 \cdot 13^3$</u>.

(D) It is now easy to verify that an O-motion is afforded by a cascade of ν-motions for R_3, R_2, R_1, R_4, R_5 in the order named; where (R_3, R_2, R_1) move as described in (A) while R_4 and R_5 go through 26-motion, which is simple rotation, with R_4 taking a step only every fourth unit of time (namely, when R_2 and R_1 are both stepping onto even-numbered points) and R_5 steps every 104^{th} time, when R_4 is stepping into its 0^{th} position. The O-motion cycles can be designated as in the case of R_3, R_2, R_1 alone, by Γ_{xjk} $(x = 0, 1, \cdots, 25;\ j = 0, 1, \cdots, 12;$ $k = 0, 1, \cdots, 12)$. Each O-motion cycle for the entire device is of length $2^4 \cdot 13^2$ and there are $2 \cdot 13^3$ of them. There are $2^4 \cdot 13^2$ cycles C_{ist} $(i = 1, 2, 3, 4;\ s = 0, 1, \cdots, 25;\ t = 0, 1, \cdots, 25)$ each of length $2 \cdot 13^3$. Each two cycles Γ_{xjk} and C_{ist} have a single common point.

5. <u>Comment on orthogonal motions</u>

It would be satisfying if our orthogonal motion had the property that every cycle Γ intersected all the cycles C in

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the same order and conversely. We could then represent the Γ's and C's as a simple reticulation of the torus by transverse and meridial circles. Unfortunately, O-motion lacks this property and would have to be complicated in a fairly ghastly manner to achieve it.

To gain further insight into the question just formulated, first consider the following comments.

(A) The proof that the points on $\Gamma_{oo\cdots o}$ are on distinct cycles $C_{x_2\cdots x_N}$ is most simply given by a recurrent argument, depending on the facts that: (1) If a two-wheel device (w_1, w_2) has ν cycles, then the points $(0, 0), (0, 1), \cdots, (0, \nu\text{-}1)$ are all on different cycles. (2) An N-wheel device can be analyzed into a sequence of two-wheel devices.

(B) The same argument carries over to prove that the points on $\Gamma_{j_1\cdots j_N}$ are on distinct cycles $C_{x_2\cdots x_N}$.

(C) There are exactly enough points on $\Gamma_{j_1\cdots j_N}$ to account for all the cycles $C_{x_2\cdots x_N}$.

(D) All the cycles $\Gamma_{j_1\cdots j_N}$ account for all settings of the machine.

These four statements constitute the heart of our argument.

To appreciate the complications of going further, we will formulate the t^{th} point on cycle $C_{x_2 x_3 \cdots x_N}$ in the special case where, for each $k = 2, 3, \cdots, N$, the number of notches on w_{k-1} is a multiple of ν_k.

For the position numbers $0, 1, \cdots, m_{k-1} - 1$ on w_{k-1} we define a function e_k as follows:

$e_k(0) = 0$

$e_k(x) =$ the number of notches at points $(0, 1, \cdots, x-1)$, $e_k(x)$ being reduced mod ν_k to one of the numbers $(0, 1, \cdots, \nu_k - 1)$.

Let $(0, x_2, x_3, \cdots, x_N)$ be the initial point, at time $t = 0$, on $C_{x_2 \cdots x_N}$, and let $\left(t, x_2^*(t), x_3^*(t), \cdots, x_N^*(t)\right)$ be the point at time t on $C_{x_2 \cdots x_N}$. Then

$$x_2^*(t) = x_2 + e_2(t)$$

$$x_3^*(t) = x_3 + e_3\left(x_2^*(t)\right) - e_3(x_3)$$

$$\vdots$$

$$x_k^*(t) = x_k + e_k\left(x_{k-1}^*(t)\right) - e_k(x_k) \qquad (k = 3, 4, \cdots, N).$$

The functions x_k^* are to be reduced mod m_k to one of the numbers $0, 1, \cdots, m_k - 1$.

It is difficult to see which of the curves $\Gamma_{j_1 \cdots j_N}$ contains the t^{th} point on $C_{x_2 \cdots x_N}$. It would be difficult to define an orthogonal motion for which one of the cycles passes through all the t^{th} points. Finally, it would be difficult to revise the cyclic order of the point on the cycles $\Gamma_{j_1 \cdots j_N}$ to agree with any specified numbering of the cycles $C_{x_2 \cdots x_N}$.